

## AN INEQUALITY FOR MEANS WITH APPLICATIONS

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ABSTRACT. We show that an almost trivial inequality between the first and second moment and the maximal value of a random variable can be used to slightly improve deep theorems.

One often estimates individual values of a function by computing a certain means. This approach is particularly useful in situations where the conjectured maximum of a function is close to its mean, as is often the case in number theory. Here, we give a simple method which sometimes allows to improve the resulting estimates. To ease applications, we formulate our result in the language of probability theory.

**Theorem 1.** *Let  $\xi$  be a non-negative real random variable, and suppose that  $\mathbf{E}\xi = 1$ , and  $\mathbf{E}\xi^2 = a$  with  $a > 1$ . Then the probability  $P(\xi \geq a)$  is positive, and for every  $b < a$  we have*

$$\int_{|\xi| > b} \xi^2 \geq a - b.$$

*Proof.* We have

$$\int_{|\xi| \leq b} \xi^2 \leq b \int_{|\xi| \leq b} \xi \leq b \int \xi = b,$$

which implies the claimed inequality. Now suppose that  $|\xi| < a$  almost surely. Then there exists some  $\epsilon > 0$ , such that  $\int_{|\xi| \leq a-\epsilon} \xi \geq \frac{1}{2}$ , and we obtain

$$\begin{aligned} a = \int \xi^2 &= \int_{|\xi| \leq a-\epsilon} \xi^2 + \int_{|\xi| > a-\epsilon} \xi^2 \\ &\leq (a-\epsilon) \int_{|\xi| \leq a-\epsilon} \xi + a \int_{|\xi| > a-\epsilon} \xi \leq \frac{(a-\epsilon) + a}{2} < a, \end{aligned}$$

a contradiction. □

We now give three applications to quite different areas.

Our first application shows that the fourth moment of the Riemann  $\zeta$ -function is dominated by large values of  $\zeta$ , in fact, by values which are so large that the fourth moment itself cannot guarantee them to exist.

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**Corollary 1.** *There is a constant  $C$  such that for  $t > T_0(\epsilon)$  and  $H > T^{2/3} \log^C T$  we have*

$$\int_{\{t \in [T, T+H] : |\zeta(\frac{1}{2} + it)| > \frac{1}{4\pi^2} \log^{3/2} t\}} |\zeta(\frac{1}{2} + it)|^4 dt \geq \frac{1-\epsilon}{4\pi^2} T \log^4 T + \mathcal{O}(T \log^3 T).$$

*Proof.* Ingham proved

$$\int_0^T |\zeta(\frac{1}{2} + it)|^2 dt = T \log \frac{T}{2\pi} + (2\gamma - 1)T + \mathcal{O}(T^{1/2+\epsilon}),$$

and Ivic and Motohashi[2] showed that

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt = T \cdot P(\log T) + \mathcal{O}(T^{2/3} \log^C T),$$

where  $P$  is a polynomial of degree 4 with leading term  $\frac{1}{2\pi^2}$  (confer also [1]). Now apply Theorem 1 by setting

$$\xi = H \cdot |\zeta(\frac{1}{2} + it)|^2 \left( \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^2 dt \right)^{-1}$$

for  $t \in [T, T+H]$  chosen at random and  $b = \frac{1}{4\pi^2} \log^2 T$ .  $\square$

Our second application slightly improves on the approach of Szekeres and Turán [5] on the problem of Hadamard-matrices.

**Corollary 2.** *For every  $\epsilon > 0$  and  $n > n_0(\epsilon)$  there exists a skew-symmetric  $n \times n$ -matrix  $A$  with entries  $\pm 1$  satisfying*

$$|\det A| > \left( \frac{n}{64\pi e^5} \right)^{1/4} e^{\sqrt{n}} \sqrt{n!}.$$

*Proof.* Let  $A$  be a random skew-symmetric  $n \times n$ -matrix with entries  $\pm 1$ , and let  $s_k(n)$  be the  $k$ -th mean of the determinant of  $A$ . Szekeres[4] showed that

$$\begin{aligned} s_1(n) &\sim \frac{1}{\sqrt[4]{8\pi e n}} e^{\sqrt{n}} \sqrt{n!}, \\ s_2(n) &\sim \frac{1}{\sqrt[4]{32\pi e^3}} e^{2\sqrt{n}} \sqrt{n!}. \end{aligned}$$

Our claim now follows by applying our theorem to  $\frac{\det A}{s_1(n)}$ .  $\square$

Our last result improves on the work of Kerov and Vershik [3] concerning the largest degree of an irreducible character of the symmetric group.

**Corollary 3.** *Let  $\epsilon > 0$  be given. Then for every  $n > n_0(\epsilon)$  there exists an irreducible character  $\chi$  of  $S_n$  with  $\chi(1) > (1 - \epsilon) e^{1/4} \sqrt{\pi n} e^{-\sqrt{n}} \sqrt{n!}$ .*

*Proof.* All irreducible complex representations of  $S_n$  can be realized over  $\mathbb{R}$ , thus

$$\sum_{\chi} \chi(1) = \#\{\pi \in S_n : \pi^2 = \text{id}\} \sim \frac{e^{\sqrt{n} - \frac{1}{4}}}{2\sqrt{\pi n}} \sqrt{n!},$$

whereas the orthogonality relation implies  $\sum_{\chi} \chi(1)^2 = n!$ . Finally, the number of irreducible characters equals the number  $p(n)$  of partitions of  $n$ , for which we have the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}.$$

Define a random variable  $\xi$  as  $\frac{\chi(1)}{\sqrt{n!}}$ , where  $\chi$  is chosen at random among all irreducible characters, where each character has the same probability. Then we obtain

$$\begin{aligned} \mathbf{E}\xi &\sim \frac{2\sqrt{3n}}{e^{1/4}\sqrt{\pi}} \exp\left((1 - \pi\sqrt{2/3})\sqrt{n}\right) \\ \mathbf{E}\xi^2 &\sim 4n\sqrt{3} \exp\left(-\pi\sqrt{2n/3}\right), \end{aligned}$$

and our claim follows.  $\square$

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